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# Classification of the Maximal Cliques of Size $\geqslant q+4$ in the Quadratic Forms Graph in Odd Characteristic

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Let $V$ denote a $d$-dimensional vector space over $\mathbb{F}_q$. Associated to $V$ is a distance-regular graph Quad$(d, q)$, the vertex set of which consists of all quadratic forms on $V$ and the edges $(x, y)$ of which are defined by the property $rk(x-y)=1$ or 2. In [8] it is shown that there are three distinguished classes of maximal cliques of respective sizes $q^d$, $q^d$ and $q^3$, and that all remaining maximal cliques $M$ satisfy $|M| \leqslant q^2+q+2$. In the present work, we obtain a strong refinement of this result when $q \geqslant 5$ is odd, providing a complete classification of all maximal cliques of size at least $q+4$.

## 1. INTRODUCTION

Let $V$ denote a $d$-dimensional vector space over the field $\mathbb{F}_q$ of $q$ elements. A quadratic form $x$ on $V$ is a map $x: V \to \mathbb{F}_q$ such that, for every $u, v \in V$ and $a, b \in \mathbb{F}_q$,

$$x(au+bv) = a^2x(u) + b^2x(v) - abB'_x(u, v)$$

for some bilinear form $B'_x$ on $V$. When $\mathbb{F}_q$ has odd characteristic we can alter our choice of form slightly to that of $B_x = (1/2)B'_x$. For our purposes $B_x$ is preferred and we shall use it exclusively. We denote by Rad$(B_x)$ and Rad$(x)$ the respective radicals of $B_x$ and $x$, i.e.

$$\mathrm{Rad}(B_x) = \{u \in V \mid B_x(u, v) = 0 \text{ for all } v \in V\},$$
$$\mathrm{Rad}(x) = \{u \in \mathrm{Rad}(B_x) \mid x(u) = 0\}.$$

The rank $rk(x)$ of $x$ is defined, as usual, to be the dimension of the quotient space $V/\mathrm{Rad}(x)$.

In this paper we are interested in a certain two-parameter family of graphs, the members Quad$(d, q)$ of which are called quadratic forms graphs. We shall address the problem of determining the maximal clique structure of Quad$(d, q)$ for fixed $d$ and $q$, with $d \geqslant 3$ and $q \geqslant 5$, $q$ odd. (For $d \leqslant 2$, Quad$(d, q)$ trivializes to a complete graph and so is of no general interest.) The graph $Q = $ Quad$(d, q)$ is defined presently.

The vertex set $V(Q)$ is precisely the set of all quadratic forms on a $d$-dimensional vector space $V$ over $\mathbb{F}_q$. Two vertices (i.e. forms) $x$ and $y$ are said to be adjacent provided that the rank $rk(x-y)$ of their difference is 1 or 2. The graph $Q$ was shown to be distance-regular by Egawa [7].

When the characteristic of $\mathbb{F}_q$ is odd, the quadratic form $x$ is uniquely determined by the bilinear form $B_x$ and the radicals Rad$(x)$ and Rad$(B_x)$ are equal. We define a graph $\Gamma$, the vertex set of which consists of all symmetric $d \times d$ matrices $M$ with entries in $\mathbb{F}_q$, and the edge set of which consists of the pairs $(M_1, M_2)$ which satisfy $rk(M_1 - M_2) = 1$ or 2. Clearly, $\Gamma$ is graph isomorphic to $Q$ iff $\mathbb{F}_q$ had odd characteristic, the canonical isomorphism being given by $x \to M_x$, where $M_x$ is the matrix representation of $B_x$ relative to a fixed choice of basis. As we are concerned only with the odd characteristic case, we shall use this fact freely (and implicitly) throughout the paper.

We first introduce some terminology, much of which we shall adopt from [8].

For $x_0 \in V(Q)$, define $C(x_0) = \{x \in V(Q) \mid rk(x - x_0) \leqslant 1\}$. We shall refer to $C(x_0)$ as



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a clique of type 1. For any $(d-1)$-dimensional subspace $W$ of $V$, define $C(W, x_0) = \{x \in V(Q) \mid x|_W = x_0|_W\}$, where $x|_W$ denotes the restriction map of $x$ to $W$. We call $C(W, x_0)$ a clique of type 2. A clique of type 1 or 2 shall be termed a grand clique. Grand cliques have size $q^d$. Finally, let $W'$ be any $(d-2)$-dimensional subspace of $V$. We define $C(W', x_0) = \{x \in V(Q) \mid \mathrm{Rad}(x - x_0) \supseteq W'\}$ and call it a cubic clique. Cubic cliques have size $q^3$. In [8] it is shown that grand cliques and cubic cliques are maximal; in fact a stronger result is proved:

[THEOREM 14]. *Let $M$ be a maximal clique of* Quad$(d, q)$. *Then either $M$ is grand or cubic, or* $|M| \leq q^2 + q + 2$.

In the present work, we establish the following refinement of Theorem 14 of [8]:

COROLLARY 6.5. *Let $M$ be a maximal clique of* Quad$(d, q)$ *where $q \geq 5$ is odd and $|M| \geq q + 4$. Then $M$ is either grand, cubic, quadratic or linear. Moreover, all grand, cubic, quadratic and linear cliques are in fact maximal of respective sizes $q^d$, $q^3$, $q^2 + 1$ and $3q - \varepsilon$, where $\varepsilon \in \{0, 1, 2, 3, 5\}$.*

Definitions for quadratic and linear cliques appear in Sections 4 and 5, respectively.

The starting point of our work is the observation, recorded in [8], that any non-grand, non-cubic maximal clique containing 0, $x$ and $y$, where $rk(x) = rk(y) = 2$ and $\mathrm{Rad}(x) = \mathrm{Rad}(y)$, must in turn be contained in the union of two distinct intersecting grand cliques (Lemma 2.11). This observation is used, by means of Lemma 3.5, to prove that any non-grand, non-cubic maximal clique of size at least $q + 4$ is so contained (Theorem 6.1). We then show that any such maximal clique is either quadratic or linear. The classification is completed by showing all quadratic and linear cliques are maximal. (Grand and cubic cliques are maximal by [8].)

The case $q = 3$ seems to require a different treatment, due to several unwelcomed (though hardly unexpected) degeneracies that arise in the graph Quad$(d, 3)$, resulting from its relatively small size. In particular, Corollary 6.5 does not hold in this case. Further work needs to be done here.

The even characteristic case will be treated in a sequel.

The balance of the paper is organized as follows. In Section 2 we derive several preliminary results, many of which follow easily from [8]. Section 3 embodies the truly quantitative aspects of the problem, the lemmas appearing there being overtly technical in nature. By resorting repeatedly to judicious choices of bases, we are able to obtain several useful relations among the entries of matrices which represent forms of prescribed type. Sections 4 and 5 are devoted, respectively, to the quadratic and linear cliques, which (along with the grand and cubic cliques of [8]) occupy a central position in our classification. Our main result appears in Section 6. In Section 7, we discuss a connection between the graph Quad $(d, q)$ and the symplectic dual polar graph $C_d(q)$.

## 2. PRELIMINARY LEMMAS

We open this section with some brief remarks, the purpose of which is to justify certain assumptions we shall make throughout the paper.

Let $K$ be an arbitrary set of forms in Quad$(d, q)$. For $x$ a fixed form in Quad$(d, q)$, we define $K - x$ to be the set of vertices given by $K - x = \{y - x \mid y \in K\}$. It is clear that the induced graph on $K - x$ is isomorphic to that induced on $K$, so that $K - x$ induces a maximal clique iff $K$ does. (In what follows we shall often abuse terminology, speaking of 'the graph $K$' rather than the more mathematically precise 'induced graph on $K$'.) We define as 'translation by $x$' (or simply 'translation') the operation by which one obtains $K-x$ from $K$. It is rather obvious that, in classifying maximal cliques of a prescribed type, one may assume that the cliques in question contain the zero form; indeed, this is tantamount to simply translating a given clique by any of its members. In the interest of clarity, we shall always emphasize when this assumption is being made.

For $C$ a grand clique containing 0, $C \neq C(0)$, there is a unique $(d-1)$-dimensional subspace $W(C)$ of $V$ associated to $C$ in a natural way. For $C$ of type 1, i.e. $C = C(x_0)$ for some rank one form $x_0$, $W(C)$ is simply $\text{Rad}(x_0)$. For $C$ of type 2, i.e. $C = C(W, 0)$, we take $W(C)$ to be the $(d-1)$-dimensional subspace $W$ of $V$ on which all members of $C$ vanish.

Our first lemma (the proof of which can be found in [8]) allows us to make an extremely important and useful reduction.

LEMMA 2.1. *Let $M$ be a non-grand maximal clique containing* 0. *Then there exists a $(d-3)$-dimensional subspace $W$ of $V$ such that for every $x \in M$, we have* $\text{Rad}(x) \supseteq W$.

Suppose then that $M$ is such a clique, and let $W'$ be a complement to $W$ in $V$. Since the radical of each form $x$ in $M$ contains $W$, no information is lost if we restrict our attention to the (three-dimensional) space $W'$. Thus we may assume from the outset that $\dim(V) = 3$ and, as a consequence, that the subspaces $W(C)$ have dimension two. For emphasis we make the following remark.

REMARK. For the balance of this paper $V$ shall be a three-dimensional vector space over $\mathbb{F}_q$.

Our next result also derives its proof from [8]. The reader will observe from (i) that $W(C) = \text{Rad}(x) \oplus \text{Rad}(y)$ whenever $x$ and $y$ are rank two forms in $C$ with $\text{Rad}(x) \neq \text{Rad}(y)$.

LEMMA 2.2. *Let $x$ and $y$ be forms, $C \neq C(0)$ a grand clique containing* 0, *and $U$ a subspace of $V$. Then the following hold*:
(i) *if $x \in C$, $x \neq 0$, then* $\text{Rad}(x) \subseteq W(C)$;
(ii) *if $U \subseteq \text{Rad}(x) \cap \text{Rad}(y)$ then $U \subseteq \text{Rad}(x+y)$*;
(iii) *let $rk(x) = 2$ and $rk(z) = 1$. Then* $\text{Rad}(x) \subseteq \text{Rad}(z)$ *iff $x$ is adjacent to $z$.*

Relative to a fixed choice of basis for $V$, we can define the restricted determinant $\det^*(x)$ of any non-zero form $x$ on $V$ to be the determinant of $x|_W$, where $W$ satisfies $V = W \oplus \text{Rad}(x)$. The value $\det^*(x)$ is not invariant under change of basis; luckily, our analysis is only concerned with whether or not $\det^*(x)$ is a square in $\mathbb{F}_q$. As this latter property is invariant under basis change, it will not be necessary for us to stipulate bases in what follows. We call a form $x$ elliptic if $\det^*(x)$ is a square and hyperbolic otherwise.

LEMMA 2.3. *Let $C$ and $K$ be distinct grand cliques with $C \cap K$ non-empty. Then $|C \cap K| = q-1$, $q$ or $q+1$. If $C$ or $K$ is of type* 2 *then $|C \cap K| = q$. If $C$ and $K$ are both type* 1, *translate by a form $z$ in $C \cap K$ to obtain $0 \in (C-z) \cap (K-z)$ and choose forms $x_0, y_0$ such that $C - z = C(x_0)$, $K - z = C(y_0)$. Then*:
(i) *$|C \cap K| = q$ if $0 \in \{x_0, y_0\}$ or* $\text{Rad}(x_0) = \text{Rad}(y_0)$. *In this case, $rk(x-y) \leqslant 1$ for all $x, y \in C \cap K$.*

(ii) $|C \cap K| = q - 1$ *if* $\mathrm{Rad}(x_0) \neq \mathrm{Rad}(y_0)$ *and if* $x_0$, $y_0$ *are both elliptic or both hyperbolic.*
(iii) $|C \cap K| = q + 1$ *if* $\mathrm{Rad}(x_0) \neq \mathrm{Rad}(y_0)$ *and one of* $x_0, y_0$ *is elliptic, the other hyperbolic.*

Proof. A complete proof appears in Theorem 9 and Corollary 10 of [8]. □

Lemma 2.4. *Given a two-dimensional subspace* $W$ *of* $V$, *and a rank two form* $x$ *with* $\mathrm{Rad}(x) \subseteq W$, *there exists a unique grand clique* $C$ *for which* $\{0, x\} \subseteq C$ *and* $W(C) = W$. *Thus a fixed rank two form lies in precisely* $q + 1$ *grand cliques which contain* 0.

Proof. Choose a basis $\{v_1, v_2, v_3\}$ for $V$ such that $W = \langle v_2, v_3 \rangle$ and $\mathrm{Rad}(x) = \langle v_2 \rangle$. Then the symmetric bilinear form $B_x$ associated with $x$ has matrix representation

$$M_x = \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ b & 0 & c \end{pmatrix}.$$

If $c = 0$ then $\{0, x\} \subseteq C$ and $W = W(C)$ for $C = C(W, 0)$. Clearly, a grand clique $K$ of type 2 is uniquely determined by $W(K)$, so that $C = C(W, 0)$ is the unique type 2 clique having these properties. But there can exist no rank one form $x_0$ for which $\{0, x\} \subseteq C(x_0)$ and $W = \mathrm{Rad}(x_0)$. Indeed, as $rk(x) = 2$, we must have $b \neq 0$, whence $rk(x - x_0) \neq 1$. This proves the lemma in the case $c = 0$.

Suppose then that $c \neq 0$. Then $x \notin C(W, 0)$, so there exist no type 2 cliques satisfying the desired properties. Let $d = a - b^2/c$. (Note that $d \neq 0$ as $rk(x) = 2$.) Then one easily checks that $C(z)$ is the unique type 1 clique satisfying $\{0, x\} \subseteq C(z)$ and $W = W(C(z))$, where the associated bilinear form $B_z$ for $z$ has representation given by

$$M_z = \begin{pmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The final statement of the lemma is obvious as there are precisely $q + 1$ two-dimensional subspaces of $V$ which contain a fixed one-dimensional subspace. □

Lemma 2.5. *Let* $C$ *and* $K$ *be distinct grand cliques containing* 0, $C \neq C(0) \neq K$. *If* $W(C) = W(K)$, *then all non-zero forms in* $C \cap K$ *are of rank one and have radical equal to* $W(C)$. *If* $W(C) \neq W(K)$, *then all non-zero forms are of rank two and have radical equal to* $W(C) \cap W(K)$. *Moreover, all non-zero forms in* $C \cap C(0)$ *are of rank one and have radical equal to* $W(C)$.

Proof. Let $z$ be an arbitrary non-zero form in $C \cap K$ and suppose that $W(C) = W(K)$. As $0 \in C \cap K$, $rk(z) \leq 2$. But $rk(z) = 2$ implies $C = K$ by Lemma 2.4. Thus $rk(z) = 1$ as asserted. Conversely, if $W(C) \neq W(K)$ then $\mathrm{Rad}(z) \subseteq W(C) \cap W(K)$, by Lemma 2.2(i), whence $rk(z) = 2$. Thus $\mathrm{Rad}(z) = W(C) \cap W(K)$ for all non-zero $z$ in $C \cap K$. Finally, all non-zero forms in $C(0)$ have rank one and we therefore have $\mathrm{Rad}(z) = W(C)$ for any such form $z$ in $C$. □

Lemma 2.6. *Let* $x$ *and* $y$ *be adjacent rank two forms satisfying* $\mathrm{Rad}(x) \neq \mathrm{Rad}(y)$. *Then there is a unique grand clique containing* $\{0, x, y\}$.

Proof. By Theorem 6 of [8], there exists a grand clique $C$ containing $\{0, x, y\}$. Suppose $\{0, x, y\}$ lies in $C'$ as well. By Lemma 2.2(i), we obtain $\mathrm{Rad}(x) \oplus \mathrm{Rad}(y) \subseteq$

$W(C)\cap W(C')$, whence $W(C)=W(C')$ by a simple dimension argument. By Lemma 2.4, we have $C=C'$ as desired.

REMARK. We shall denote by $[0,x,y]$ the unique grand clique determined by $x$ and $y$ in the sense of Lemma 2.6.

For any set $J$ of forms in $\mathrm{Quad}(d,q)$, we define $N(J)$ to be the set of common neighbors of all members of $J$. For $H$, $J$ sets of forms in $\mathrm{Quad}(d,q)$, we further define $N_H(J)=N(J)\cap H$. Finally, we shall always write $N_H(x_1,\ldots,x_m)$ in lieu of the more cumbersome $N_H(\{x_1,\ldots,x_m\})$.

LEMMA 2.7. *Let $x$ and $y$ be adjacent rank two forms with* $\mathrm{Rad}(x)\neq\mathrm{Rad}(y)$ *and let* $C=[0,x,y]$. *Then*

$$C=\{0,x,y\}\cup\{z\in N(0,x,y)\mid \mathrm{Rad}(z)\subseteq\mathrm{Rad}(x)\oplus\mathrm{Rad}(y)\}.$$

PROOF. If $z$ is a non-zero form in $C$ then, by Lemma 2.2(i), $\mathrm{Rad}(z)\subseteq W(C)\subseteq \mathrm{Rad}(x)\oplus\mathrm{Rad}(y)$. Conversely, let $z\in N(0,x,y)$ with $\mathrm{Rad}(z)\subseteq\mathrm{Rad}(x)\oplus\mathrm{Rad}(y)$. If $rk(z)=1$ then clearly $z\in C$; so assume $rk(z)=2$ and, without loss, that $\mathrm{Rad}(z)\neq \mathrm{Rad}(x)$. Then $W(C)=\mathrm{Rad}(x)\oplus\mathrm{Rad}(y)=\mathrm{Rad}(x)\oplus\mathrm{Rad}(z)=W(K)$, where $K=[0,x,z]$. But $x\in C\cap K$, whence $C=K$ by Lemma 2.4. In particular, $z\in C$, proving the reverse inclusion. □

LEMMA 2.8. *Let $S$ be an arbitrary clique containing* 0 *and a rank two form. Then $S$ lies in a cubic clique iff all rank two forms in $S$ have the same radical. In particular, all rank two forms of a cubic clique containing* 0 *have the same radical.*

PROOF. Let $T$ be a cubic clique containing $S$. By definition there is a one-dimensional subspace $U$ and a form $x_0$ such that $T=\{x\mid \mathrm{Rad}(x-x_0)\supseteq U\}$. As $0\in T$ we have $\mathrm{Rad}(x_0)\supseteq U$. Let $x$ be any rank two form in $T$. Then $\mathrm{Rad}(x-x_0)\supseteq U$ so that $\mathrm{Rad}(x)=\mathrm{Rad}((x-x_0)+x_0)\supseteq U$ by Lemma 2.2(ii). By dimension we have $\mathrm{Rad}(x)=U$, whence all rank two forms of $T$ (so also of $S$) have the same radical.

Conversely, suppose all rank two forms of $S$ have the same radical and let $x$ be one of them. As $0\in S$ all forms of $S$ are of rank at most two. But for any rank one form $z$ in $S$ we clearly have $\mathrm{Rad}(z)\supseteq\mathrm{Rad}(x)$ by Lemma 2.2(iii). Thus $S$ is contained in the cubic clique $C(0,\mathrm{Rad}(x))$, and the lemma is proved. □

LEMMA 2.9. *Let $C$ be a clique containing* 0 *and three rank two forms $x, y, z$ with* $V=\mathrm{Rad}(x)\oplus\mathrm{Rad}(y)\oplus\mathrm{Rad}(z)$. *Then $C$ is not contained in any grand or cubic clique.*

PROOF. Immediate from Lemmas 2.2 and 2.8. □

LEMMA 2.10. *Let $M$ be a maximal clique containing* 0 *which is neither grand nor cubic. Then $M$ contains no rank one form. As a consequence, any distinct forms $x, y\in M$ satisfy $rk(x-y)=2$.*

PROOF. By way of contradiction, assume $w\in M$ with $rk(w)=1$. If $M$ has no rank two forms, then trivially $M$ lies in the grand clique $C(0)$. If all rank two forms in $M$ have the same radical, then by Lemma 2.8, $M$ lies in a cubic clique. As these are obvious contradictions, we conclude that $M$ contains two rank two forms $x$ and $y$ with $\mathrm{Rad}(x)\neq\mathrm{Rad}(y)$. We therefore have $\mathrm{Rad}(x)\oplus\mathrm{Rad}(y)=\mathrm{Rad}(w)$. We claim that

$\mathrm{Rad}(z) \subseteq \mathrm{Rad}(x) \oplus \mathrm{Rad}(y)$ for all non-zero $z$ in $M$. Indeed, as $M$ contains 0, all forms $z$ in $M$ must have rank at most two. As already observed, $rk(z)=2$ implies $\mathrm{Rad}(z) \subseteq \mathrm{Rad}(w) = \mathrm{Rad}(x) \oplus \mathrm{Rad}(y)$. If $rk(z)=1$ then $\mathrm{Rad}(z)$ contains both $\mathrm{Rad}(x)$ and $\mathrm{Rad}(y)$, whence $\mathrm{Rad}(z) = \mathrm{Rad}(x) \oplus \mathrm{Rad}(y)$, proving the claim. But then $M = [0, x, y]$ by Lemma 2.7, i.e. $M$ is a grand clique. This being our final contradiction, the lemma is proved. □

LEMMA 2.11. *Let $M$ be a maximal clique containing* 0 *and two rank two forms $x$ and $y$ with* $\mathrm{Rad}(x) = \mathrm{Rad}(y)$. *Then either $M$ is a grand or cubic clique, or there exist distinct grand cliques $C$ and $K$ with $M \subseteq C \cup K$ and $\{0, x, y\} \subseteq C \cap K$.*

PROOF. The proof follows at once from (iii), (iv), and (v) in the proof of Theorem 14 of [8]. □

## 3. TECHNICAL LEMMAS

In this section we focus our attention on the following general situation. Let $C$ be a grand clique and $(x, y)$ an edge totally external to $C$, i.e. $x, y \notin C$ with $x \in N(y)$. Roughly speaking, we are interested in determining necessary conditions for membership to $N_C(x, y)$ under suitable restrictions on $x$ and $y$. To be precise, let us formulate the following set of hypotheses for use in Lemmas 3.1–3.4.

EXTERNAL EDGE HYPOTHESES. Let $C_1 \neq C(0)$ be a grand clique containing 0 and let $C_2 = [0, x, y]$, where $x, y \notin C_1$ are adjacent rank two forms with distinct radicals which satisfy $rk(x-y)=2$. Suppose neither of $\mathrm{Rad}(x)$, $\mathrm{Rad}(y)$ is contained in $W(C_1)$ and let $z$ be an arbitrary form in $N_L(x, y)$, where $L = C_1 \backslash C_2$.

LEMMA 3.1. *Assume the external edge hypotheses with $C_1$, $C_2$ both of type 2, say $C_1 = C(W_1, 0)$ and $C_2 = C(W_2, 0)$, where $W_1$ and $W_2$ are distinct two-dimensional subspaces of $V$. Then there exists a basis for $V$ with respect to which $B_x$, $B_y$ and $B_z$ are represented by the matrices*

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & b \end{pmatrix}, \qquad M_y = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ c & d & e \end{pmatrix}, \qquad M_z = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}.$$

*Furthermore, one of the following relations holds among the entries*:
(i) $\beta = 2cd/(b-e)$, *which occurs if* $\mathrm{Rad}(x-y) \subseteq W_1$;
(ii) $\gamma = A\beta + B$, *which occurs otherwise.*
*In* (ii), $A = (a^2e - d^2b)/2ad(a-d)$ *and* $B = ac/(a-d)$.

PROOF. The desired basis is easily obtained by choosing respective bases $\{v_1, v_2\}$ for $W_2$ and $\{v_2, v_3\}$ for $W_1$, with $v_1$ additionally chosen in $\mathrm{Rad}(x)$. Having done this, we observe that $z$ lies in $N_L(x)$ iff $\det(M_z - M_x) = 0$, i.e. iff

$$\alpha = (\beta^2 b - 2\beta\gamma a)/a^2, \tag{1}$$

where $a \neq 0$ as $rk(x)=2$. Moreover, $\beta \neq 0$, as otherwise we would have $z \in C_2$. Similarly, $z \in N_L(y)$ iff

$$\alpha = (\beta^2 e - 2\beta d(\gamma - c))/d^2, \tag{2}$$

where $d \neq 0$ as, otherwise, we would have $\mathrm{Rad}(y) \subseteq W_1$.

Equating the expressions for $\alpha$ in (1) and (2) yields

$$d^2(\beta^2 b - 2\beta\gamma a) = a^2(\beta^2 e - 2\beta d(\gamma - c)). \tag{3}$$

If $\mathrm{Rad}(x-y) \subseteq W_1$ then clearly $a = d$, whence (3) simplifies to $\beta b = \beta e + 2dc$. Now $b = e$ implies $c = 0$, a contradiction as $\mathrm{Rad}(x) \neq \mathrm{Rad}(y)$. Thus $b \neq e$ and $\beta = 2cd/(b-e)$ as claimed. On the other hand, if $\mathrm{Rad}(x-y)$ is not contained in $W_1$ then $a \neq d$. We leave it to the reader to verify that solving (3) for $\gamma$ yields the desired result when $a \neq d$. □

LEMMA 3.2. *Assume the external edge hypotheses with $C_1$ of type 2 and $C_2$ of type 1, say $C_1 = C(W, 0)$ and $C_2 = C(x_0)$, where $W$ is a two-dimensional subspace of $V$ and $x_0$ is a rank one form. Then there exists a basis for $V$ with respect to which $B_x$, $B_y$, $B_z$ and $B_{x_0}$ are represented by the matrices*

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & (b^2/a)+s \end{pmatrix}, \qquad M_y = \begin{pmatrix} c^2/d & c & ce/d \\ c & d & e \\ ce/d & e & (e^2/d)+s \end{pmatrix},$$

$$M_z = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}, \qquad M_{x_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \end{pmatrix}.$$

*Furthermore, one of the following relations holds among the entries:*
(i) *$\beta = 2ac/(a-d)$, which occurs when* $\mathrm{Rad}(x-y) \subseteq W$;
(ii) *$\gamma = A\beta + B$, which occurs otherwise.*
*In* (ii), *$A = [sad(a-d) + a^2e^2 - b^2d^2]/2ad(ae-bd)$ and $B = acs/(bd-ae)$.*

PROOF. To obtain the desired basis, choose $\{v_1, v_2\}$ a basis for $\mathrm{Rad}(x_0)$ with $v_1 \in \mathrm{Rad}(x)$, and $\{v_2, v_3\}$ a basis for $W$. The form of the entries in $M_x$, $M_y$ follows from the fact that $rk(x-x_0) = rk(y-x_0) = 1$. (Observe that $a, d \neq 0$ as $\mathrm{Rad}(x)$, $\mathrm{Rad}(y)$ are not contained in $W$.) Now $z \in N(x)$ iff $\det(M_x - M_z) = 0$, i.e. iff

$$\alpha = (2ab\beta\gamma - a^2\gamma^2 - a\beta^2 s - \beta^2 b^2)/a^2 s. \tag{4}$$

Similarly, $z \in N(y)$ is equivalent to

$$\alpha = (2\beta\gamma de + 2\beta cds - \beta^2 ds - \beta^2 e^2 - \gamma^2 d^2)/d^2 s. \tag{5}$$

Equating the expressions for $\alpha$ in (4) and (5) gives

$$2\gamma ad(ae - bd) = \beta sad(a-d) + \beta(a^2e^2 - b^2d^2) - 2a^2cds. \tag{6}$$

If $\mathrm{Rad}(x-y) \subseteq W$ then $\mathrm{Rad}(x-y) = \langle (e/d)v_2 - v_3 \rangle$, whence $ae = bd$. From this and (6) we obtain $\beta = 2ac/(a-d)$. (Observe that $a \neq d$, as otherwise we would have $c = 0$ which contradicts the fact that $\mathrm{Rad}(x) \neq \mathrm{Rad}(y)$.) Conversely, if $\mathrm{Rad}(x-y)$ is not contained in $W$, then it follows that $ae \neq bd$ and the corresponding solution for $\gamma$ is obtained explicitly from (6). We leave the verification of this to the reader. □

LEMMA 3.3. *Assume the external edge hypotheses with $C_1$ of type 1 and $C_2$ of type 2, say $C_1 = C(z_0)$ and $C_2 = C(W, 0)$, where $z_0$ is a rank one form and $W$ is a two-dimensional subspace of $V$. Then there exists a basis for $V$ with respect to which $B_x$,*

*$B_y$, $B_{z_0}$ and $B_z$ are represented by the matrices*

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & a & b \end{pmatrix}, \qquad M_y = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ c & d & e \end{pmatrix}, \qquad M_{z_0} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

*and*

$$M_z = \begin{pmatrix} f + (\alpha^2/\beta) & \alpha & \alpha\gamma/\beta \\ \alpha & \beta & \gamma \\ \alpha\gamma/\beta & \gamma & \gamma^2/\beta \end{pmatrix} \qquad (\beta \neq 0).$$

*Furthermore, the entries satisfy the relation*

$$\alpha = A\beta + B\gamma,$$

*where $A = (a^2ef + a^2c^2 - bd^2f)/2a^2cd$ and $B = f(d-a)/ac$.*

Proof. We choose our basis $\{v_1, v_2, v_3\}$ as follows. Let $v_1$ be any non-zero vector in $\mathrm{Rad}(x)$; similarly, choose $v_2 \in \mathrm{Rad}(z_0) \cap W$ and $v_3 \in \mathrm{Rad}(z_0)$ independent of $v_2$. Clearly, $M_x$, $M_y$ and $M_{z_0}$ have the form given in the lemma statement. The form for $M_z$ is now derived from the fact that $rk(z - z_0) = 1$. (Observe that $\beta \neq 0$ as otherwise $z \in C_2$ by Lemma 2.7.) From the condition $\det(M_z - M_x) = 0$, one now derives the equation

$$\beta^2 bf + \beta a^2 f - 2\beta\gamma af + \alpha^2 a^2 = 0. \tag{7}$$

Similarly, $\det(M_z - M_y) = 0$ yields

$$\beta^2 ef + \beta^2 c^2 + \beta d^2 f - 2\alpha\beta cd - 2\beta\gamma df + \alpha^2 d^2 = 0. \tag{8}$$

Let (7′) and (8′) be the equations one obtains by multiplying (7) by $d^2$ and (8) by $a^2$, respectively. Subtracting (8′) from (7′) and solving for $\alpha$ gives the equation which appears in the lemma statement. □

Lemma 3.4. *Assume the external edge hypotheses with $C_1$ and $C_2$ both of type 1, say $C_1 = C(z_0)$ and $C_2 = C(x_0)$, where $z_0$, $x_0$ are rank one forms. Then there exists a basis for $V$ with respect to which $B_x$, $B_y$, $B_{x_0}$, $B_{z_0}$ and $B_z$ are represented by the matrices*

$$M_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & g + (b^2/a) \end{pmatrix}, \qquad M_y = \begin{pmatrix} c^2/d & c & ce/d \\ c & d & e \\ ce/d & e & g + (e^2/d) \end{pmatrix},$$

$$M_{x_0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g \end{pmatrix}, \qquad M_{z_0} = \begin{pmatrix} f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_z = \begin{pmatrix} f + (\alpha^2/\beta) & \alpha & \alpha\gamma/\beta \\ \alpha & \beta & \gamma \\ \alpha\gamma/\beta & \gamma & \gamma^2/\beta \end{pmatrix} \qquad (\beta \neq 0).$$

*Furthermore, the entries satisfy the relation*

$$\alpha = A\beta + B\gamma,$$

*where*

$$A=\frac{b^2d^2f+ad^2fg+a^2c^2g-a^2e^2f-a^2dfg}{2a^2cdg}$$

*and*

$$B=(aef-bdf)/acg.$$

Proof. By choosing $\{v_1, v_2\}$ as a basis for $\mathrm{Rad}(x_0)$ and $\{v_2, v_3\}$ a basis for $\mathrm{Rad}(z_0)$, we clearly have the matrices $M_{x_0}$ and $M_{z_0}$ as given above. We may further assume that $v_1$ has been chosen in $\mathrm{Rad}(x)$, whence $M_x$ is as claimed since $rk(x-x_0)=1$. Similarly, the form for each of $M_y$ and $M_z$ is derived as in the two previous lemmas. We leave it to the reader to verify that the desired equation for $\gamma$ is obtained directly from the adjacency conditions $\det(M_z-M_x)=0$ and $\det(M_z-M_y)=0$ in a manner similar to that of Lemma 3.3. □

Lemma 3.5. *Let $C_1$ and $C_2$ be grand cliques containing 0 and let $x_1, y_1, z_1 \in C_1\backslash C_2$ and $x_2, y_2 \in C_2\backslash C_1$ be pairwise adjacent rank two forms, no two of which have the same radical. Further, assume that $rk(r-s)=2$ for $r, s \in \{x_1, y_1, z_1, x_2, y_2\}$. Then* $\mathrm{Rad}(y_1-x_1)=\mathrm{Rad}(z_1-x_1)$.

Proof. We first observe that $C_i \neq C(0)$ as $C_i$ contains rank two forms $(i=1,2)$. Also as $\mathrm{Rad}(x_i)\neq \mathrm{Rad}(y_i)$ we have $W(C_i)=\mathrm{Rad}(x_i)\oplus\mathrm{Rad}(y_i)$, whence neither of $\mathrm{Rad}(x_i)$, $\mathrm{Rad}(y_i)$ is contained in $W(C_j)$ $(1\leqslant i\neq j\leqslant 2)$. (Indeed if, for example, $\mathrm{Rad}(x_1)\subseteq W(C_2)$ then $\mathrm{Rad}(x_1)\subseteq \mathrm{Rad}(x_2)\oplus\mathrm{Rad}(y_2)$, whence $x_1\in C_2$, a contradiction.) In particular, the external edge hypotheses are satisfied. We divide the remainder of the proof into two cases which address the different situations which arise as the types of $C_1$ and $C_2$ vary.

*Case one:* $C_1$, $C_2$ as in Lemma 3.1 or Lemma 3.2. We first assume $\mathrm{Rad}(x_2-y_2)\subseteq W(C_1)$. Here we observe that $\beta$ is independent of the form chosen from $N_L(x_2, y_2)$. (Recall that $\beta=2cd/(b-e)$ in Lemma 3.1 and $\beta=2ac/(a-d)$ in Lemma 3.2, so in either case $\beta$ depends only on $x_2$ and $y_2$.) It is now immediate that $\mathrm{Rad}(y_1-x_1)=\langle v_2\rangle=\mathrm{Rad}(z_1-x_1)$ as desired. So we assume $\mathrm{Rad}(x_2-y_2)$ is not contained in $W(C_1)$. In this case we have $\gamma=A\beta+B$, where $A$, $B$ depend only on $x_2$, $y_2$, although $\gamma$, $\beta$ may well vary with the choice of form in $N_L(x_2, y_2)$. Nonetheless, it is easy to verify that $\mathrm{Rad}(y_1-x_1)=\langle Av_2-v_3\rangle=\mathrm{Rad}(z_1-x_1)$ in this case.

*Case two:* $C_1$, $C_2$ as in Lemma 3.3 or Lemma 3.4. In each of these lemmas, we observed that $\alpha$, $\beta$, $\gamma$ satisfied $\alpha=A\beta+\beta\gamma$ for certain specified constants $A$, $B$ which depended only on the forms $x_2$, $y_2$. It is a simple exercise to verify that $\langle v_1-Av_2-Bv_3\rangle$ is the common radical of $y_1-x_1$ and $z_1-x_1$. □

## 4. Quadratic Cliques

The theory developed in this section is geared toward a study of a specific class of maximal cliques, the members of which we call quadratic. Unfortunately, the nature of our classification mandates that we postpone a proof of maximality until Section 6. We nonetheless define quadratic cliques presently, in the hope that the reader will thereby obtain a more lucid understanding of the motivation underlying the results of this section. The main results, which will be used in Section 6, are Lemmas 4.5 and 4.6.

Definition (quadratic clique). Let $C$ be a grand clique containing 0, $C\neq C(0)$, and let $x$ be a rank two form with $\mathrm{Rad}(x)$ not contained in $W(C)$. Under these conditions, we call $\{x\}\cup N_C(x)$ (as well as any of its translates) a quadratic clique.

LEMMA 4.1. *Let $x$ be a rank two form. Then $\{0, x\}$ is contained in a type 2 clique iff $-\det^*(x)$ is a square.*

PROOF. Choose a basis $\{v_1, v_2, v_3\}$ for $V$ with $v_3 \in \mathrm{Rad}(x)$ and let $a = B_x(v_1, v_1)$, $b = B_x(v_1, v_2)$, $c = B_x(v_2, v_2)$. Observe that $\{0, x\}$ is in a type 2 clique $C = C(W, 0)$ iff $x|_W = 0$. Such a $W$ exists iff there exists a non-zero $v \notin \mathrm{Rad}(x)$ such that $x(v) = 0$, which we may assume has the form $v = \alpha v_1 + \beta v_2$ for some $\alpha, \beta \in F$. If $c = 0$ then $-\det^*(x)$ is a square and we may take $W = \langle v_2, v_3\rangle$. So assume $c \neq 0$. We may also assume $\alpha = 1$ as $x(v) = 0$ and $\alpha = 0$ together imply $v = 0$. Thus the existence of such a $v$ is equivalent to the existence of a solution to the equation $c\beta^2 + 2b\beta + a = 0$, which in turn is equivalent to $-(ac - b^2) = -\det^*(x)$ being a square. The result follows. □

LEMMA 4.2. *Let $x$ be a rank two form*:
(i) *If $-\det^*(x)$ is a square, then $\{0, x\}$ is contained in precisely two grand cliques of type* 2 *and $q - 1$ grand cliques of type* 1.
(ii) *If $-\det^*(x)$* is a non-square, then *$\{0, x\}$ is contained in precisely $q + 1$ grand cliques, all of type* 1.

PROOF. By Lemmas 2.4 and 4.1, it suffices to show that $\{0, x\}$ is contained in precisely two grand cliques of type 2 whenever it lies in one such clique. So assume that $x$ lies in the type 2 clique $C_1$. By an appropriate choice of basis, we may assume that $C_1 = C(0, W_1)$, where $W_1 = \langle v_2, v_3\rangle$ and $x$ has representation

$$M_x = \begin{pmatrix} a & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Letting $W_2 = \langle v_1 - (a/2b)v_2, v_3\rangle$, it is trivial to verify that $\{0, x\}$ is contained in the type 2 clique $C_2 = C(0, W_2)$. As $\langle v_2\rangle$ and $\langle v_1 - (a/2b)v_2\rangle$ are the only one-dimensional subspaces of $\langle v_1, v_2\rangle$ on which $x$ vanishes, there is no other type 2 clique which contains $\mathrm{Rad}(x)$. The result follows. □

Let $S$ denote the set of non-zero squares of $\mathbb{F}_q$ and $N$ the set of non-squares. We define the sets $A_{++}$ and $A_{-+}$ as follows:

$$A_{++} = \{a \in S \mid a + 1 \in S\}, \qquad A_{-+} = \{a \in N \mid a + 1 \in S\}.$$

The following result is due to Dickson. A proof can be found in [6].

LEMMA 4.3:

$$|A_{++}| = \begin{cases} (q-5)/4 & \text{if } -1 \in S; \\ (q-3)/4 & \text{if } -1 \in N; \end{cases}$$

$$|A_{-+}| = \begin{cases} (q-1)/4 & \text{if } -1 \in S, \\ (q-3)/4 & \text{if } -1 \in N. \end{cases}$$

LEMMA 4.4. *Let $x$ be a rank two form and consider the set $\{C(z_i)\}_{i=1,\ldots,k}$ of all type* 1 *cliques which contain $\{0, x\}$. Then either $k = q + 1$ or $k = q - 1$ and exactly half of the forms $z_i$ are elliptic.*

PROOF. Choose a basis with respect to which $x$ is represented by

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then each $z_i$ has matrix representation given by

$$M_{z_i} = \begin{pmatrix} a_i + 1 & b_i & 0 \\ b_i & \cdot\, c_i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for suitable constants $a_i$, $b_i$, $c_i$. Suppressing subscripts, we obtain the equations $(a+1)c = b^2$ and $a(c-d) = b^2$ or, equivalently, $c = -ad$ and $-ad(a+1) = b^2$.

Let $Y = \{a \mid a+1 \in S,\ -ad(a+1) \in S\}$. We use $Y$ to determine the number of elliptic forms among the $z_i$ in each of four cases.

*Case one:* $-d \in S$, $-1 \in S$. In this case $|Y| = |A_{++}| = (q-5)/4$ (Lemma 4.3). Here each scalar $a$ in $Y$ gives rise to precisely two elliptic forms, each corresponding to a fixed choice of $b$ satisfying $b^2 = -ad(a+1)$. Moreover, we obtain two additional elliptic forms which arise, correspondingly, from the choices $a=0$ and $a+1=0$. This gives a total of $2(q-5)/4 + 2 = (q-1)/2$ elliptic forms in all. By Lemma 4.2(i), there are two type 2 cliques which contain $\{0, x\}$. Thus exactly half of the type 1 forms which contain $\{0, x\}$ are elliptic as claimed.

*Case two:* $-d \in S$, $-1 \in N$. Here again $Y = A_{++}$, but this time the form corresponding to $a+1=0$ is hyperbolic. The number of elliptic forms among the $z_i$ is therefore given by $2(q-3)/4 + 1 = (q-1)/2$, which by Lemma 4.2(i) is again half of the full number $k$ of type 1 forms.

*Case three:* $-d \in N$, $-1 \in S$. This time $|Y| = |A_{-+}| = (q-1)/4$. Arguing as in case two, we count the total number of elliptic forms, obtaining $2(q-1)/4 + 1 = (q+1)/2$, which is the desired number according to Lemma 4.2(ii).

*Case four:* $-d \in N$, $-1 \in N$. Here $|Y| = |A_{-+}| = (q-3)/4$. As the forms corresponding to $a=0$ and $a+1=0$ are again both elliptic, we obtain $2(q-3)/4 + 2 = (q+1)/2$ as desired. □

LEMMA 4.5. *Let $C \neq C(0)$ be a grand clique containing* 0, *and let $x \notin C$ be a rank two form with* Rad$(x)$ *not contained in* $W(C)$. *Then* $|N_C(x)| = q^2$.

PROOF. By Lemma 2.4, $\{0, x\}$ lies in precisely $q+1$ grand cliques $C_1, C_2, \ldots, C_{q+1}$. We claim $N_C(x) = \cup(C \cap C_i)$, where the union ranges over $1 \leqslant i \leqslant q+1$. Indeed, for $y \in C \cap C_j$, $y$ is adjacent to $x$ (as they both lie in $C_j$) so that $N_C(x) \supseteq \cup(C \cap C_i)$. Conversely, if $y \in N_C(x)$ we have $rk(y) = 2$ and $\mathrm{Rad}(x) \neq \mathrm{Rad}(y)$ (by Lemma 2.2(iii) and our hypothesis on Rad$(x)$), so that $[0, x, y]$ is a grand clique containing $x$, whence $[0, x, y] = C_k$ for some $k$. Thus, clearly, $N_C(x) \subseteq \cup(C \cap C_i)$, which proves the claim. Moreover, $C \cap C_j \cap C_k = \{0\}$ for $j, k$ distinct, for if there existed a non-zero form $z \in C \cap C_j \cap C_k$ we would have $C_j = [0, x, z] = C_k$, a contradiction. This proves $|N_C(x)| = (\sum |C \cap C_i|) - q$. But Lemma 4.4, with the aid of Lemma 2.3, implies that the average value of $|C \cap C_i|$ as $i$ ranges over $1 \leqslant i \leqslant q+1$ is $q$. Thus $|N_C(x)| = (q+1)q - q = q^2$ as desired. □

LEMMA 4.6. *Let $C \neq C(0)$ be a grand clique containing* 0 *and let $x \notin C$ be a rank two form. Then $\{x\} \cup N_C(x)$ is contained in a grand or cubic clique iff* $\mathrm{Rad}(x) \subseteq W(C)$.

PROOF. If $\mathrm{Rad}(x) \subseteq W(C)$, then all rank one forms in $C$ lie in $N_C(x)$, and the result follows from Lemma 2.10. Conversely, if $\mathrm{Rad}(x)$ is not contained in $W(C)$ then, as in the proof of Lemma 4.5, we can find $y \in N_C(x)$ with $\mathrm{Rad}(y)$ equal to any prescribed one-dimensional subspace of $W(C)$. Thus $\{x\} \cup N_C(x)$ satisfies the hypothesis of Lemma 2.9 and the result follows. □

## 5. LINEAR CLIQUES

We begin by defining a class of cliques in $\mathrm{Quad}(d, q)$, the members of which, termed linear cliques, will be shown to be maximal in Section 6. First some preliminary notation is needed.

For $H$, $J$, $K$ any sets of forms in $\mathrm{Quad}(d, q)$, let $(H, J, K)$ denote the set $(H \cap J) \cup (H \cap K) \cup (J \cap K)$. The reader will readily observe that $(H, J, K)$ is a clique whenever $H$, $J$ and $K$ are.

DEFINITION (linear clique). Let $C_1$, $C_2$, $C_3$ be grand cliques with $0 \in C_1 \cap C_2$, $x_1 \in C_1 \cap C_3$, $x_2 \in C_2 \cap C_3$, $rk(x_1) = rk(x_2) = 2$, and $\mathrm{Rad}(x_i)$ not contained in $W(C_j)$ for $1 \leq i \neq j \leq 2$. We call $(C_1, C_2, C_3)$ (as well as any of its translates) a linear clique.

LEMMA 5.1. *Let $C_1$ and $C_2$ be grand cliques containing* 0 *with $W(C_1) \neq W(C_2)$. Let $x_1, y_1, x_2, y_2$ be pairwise adjacent rank two forms with $x_1, y_1 \in C_1$, $x_2, y_2 \in C_2$, and suppose that no one of* $\mathrm{Rad}(x_1)$, $\mathrm{Rad}(y_1)$, $\mathrm{Rad}(x_2)$, $\mathrm{Rad}(y_2)$ *coincides with $W(C_1) \cap W(C_2)$. Then* $\mathrm{Rad}(x_1) = \mathrm{Rad}(y_1)$ *iff* $\mathrm{Rad}(x_2) = \mathrm{Rad}(y_2)$.

PROOF. By symmetry it suffices to show that $\mathrm{Rad}(x_1) = \mathrm{Rad}(y_1)$ implies $\mathrm{Rad}(x_2) = \mathrm{Rad}(y_2)$, so assume that $\mathrm{Rad}(x_1) = \mathrm{Rad}(y_1)$ and $\mathrm{Rad}(x_2) \neq \mathrm{Rad}(y_2)$. Then, clearly, $C_2 = [0, x_2, y_2]$. Let $z \in C_1 \cap C_2$, $z \neq 0$. Then $\mathrm{Rad}(z) = W(C_1) \cap W(C_2)$; in particular $\mathrm{Rad}(z) \neq \mathrm{Rad}(x_1)$, whence $C_1 = [0, x_1, z]$. Let $M$ be a maximal clique which contains $\{0, x_1, y_1, x_2, y_2, z\}$. As $\mathrm{Rad}(x_1) + \mathrm{Rad}(x_2) + \mathrm{Rad}(y_2) = \mathrm{Rad}(x_1) \oplus W(C_2) = V$ (since $\mathrm{Rad}(x_1) \neq W(C_1) \cap W(C_2)$), we conclude from Lemma 2.9 that $M$ is neither grand nor cubic. Thus, by Lemma 2.11, there exist distinct grand cliques $C$ and $K$ with $M \subseteq C \cup K$ and $\{0, x_1, y_1\} \subseteq C \cap K$. Without loss of generality, $z \in C$, whence $C = [0, x_1, z] = C_1$. If $x_2 \in C$, then $\mathrm{Rad}(x_2) = W(C_1) \cap W(C_2)$, a contradiction. Similarly, $y_2 \notin C$. Thus $K = [0, x_2, y_2] = C_2$, whence $x_1 \in C_2$. But this implies that $\mathrm{Rad}(x_1) = W(C_1) \cap W(C_2)$, which is our final contradiction. □

LEMMA 5.2. *Let $C_1$, $C_2$, $C_3$ be grand cliques with $|C_1 \cap C_2 \cap C_3| \geq 2$. Then $(C_1, C_2, C_3)$ is contained in a cubic clique.*

PROOF. Let $I = C_1 \cap C_2 \cap C_3$. Translating if necessary, we may assume $0 \in I$. Let $z \in I$, $z \neq 0$. If $rk(z) = 1$, either $C_i = C(0)$ for some $i$ or $W(C_i) = \mathrm{Rad}(z)$ for all $i$. In any case, all non-zero forms in $(C_1, C_2, C_3)$ have radical $\mathrm{Rad}(z)$ by Lemma 2.5, whence $(C_1, C_2, C_3)$ is contained in a cubic clique. Similarly, if $rk(z) = 2$, we again have (by Lemma 2.5) that all non-zero forms in $(C_1, C_2, C_3)$ have radical $\mathrm{Rad}(z)$, and the result follows from Lemma 2.8. □

LEMMA 5.3. *Let $C_1$ and $C_2$ be grand cliques containing* 0 *and let $M$ be a maximal clique, neither grand nor cubic, contained in $C_1 \cup C_2$. Let $x_1, y_1, x_2, y_2$ be rank two forms in $M$ with $x_1, y_1 \notin C_2$ and $x_2, y_2 \notin C_1$. Then there exists a unique grand clique $C_3$ such that $M = (C_1, C_2, C_3)$.*

PROOF. Since $C_1 \cap C_2 \subseteq M$ and since all non-zero forms in $M$ must be of rank two by Lemma 2.10, we have $W(C_1) \neq W(C_2)$ by Lemma 2.5. By Lemma 2.8, $M$ contains a rank two form $z$ with $\mathrm{Rad}(z) \neq W(C_1) \cap W(C_2)$. We may assume $z \in C_1$. Then, by Lemma 2.7, no form in $M \backslash C_1$ has radical equal to $W(C_1) \cap W(C_2)$, which in turn implies that no form in $M \backslash C_2$ has radical equal to $W(C_1) \cap W(C_2)$. Thus Lemma 5.1 applies, and all forms in $M \backslash C_i$ have either common radical $\mathrm{Rad}(x_i)$ or radicals which are pairwise distinct. In the former case, the result follows from Lemma 2.7 with $C_3 = [0, x_1, x_2]$. In the latter case, the result will follow with $C_3 = [0, y_1 - x_1, x_2 - x_1] + x_1$, as we now show.

First observe that $C_3$ is well defined as $\mathrm{Rad}(y_1 - x_1) \neq \mathrm{Rad}(x_2 - x_1)$. (Indeed, we otherwise have $x_2 - x_1 \in [0, -x_1, y_1 - x_1] = C_1 - x_1$ by Lemma 2.7, an obvious contradiction.) Applying Lemma 3.5, we obtain $\mathrm{Rad}(y_1 - x_1) = \mathrm{Rad}(t - x_1)$ for all $t \in M \backslash C_2$. By Lemma 2.7, $t - x_1 \in C_3 - x_1$ for all such $t$, i.e. $M \backslash C_2 \subseteq C_3$. It therefore suffices to prove that $M \backslash C_1 \subseteq C_3$.

Therefore let $r$ be an arbitrary form in $M \backslash C_1$. Applying Lemma 2.11, there exist distinct grand cliques $K_1$ and $K_2$ which satisfy $M - x_1 \subseteq K_1 \cup K_2$ and $\{0, z - x_1, y_1 - x_1\} \subseteq K_1 \cap K_2$. We may assume that $-x_1 \in K_1$, whence $K_1 = [0, -x_1, y_1 - x_1] = C_1 - x_1$. As $x_2 \notin C_1$, it follows that $x_2 - x_1 \in K_2$, whence $K_2 = [0, x_2 - x_1, y_1 - x_1] = C_3 - x_1$. As $r \notin C_1$, we must have $r - x_1 \in K_2$, i.e. $r \in C_3$. Thus $M \backslash C_1 \subseteq C_3$, and the proof is complete. □

## 6. Classification of Maximal Cliques

Without further ado...

THEOREM 6.1. *Let $M$ be a non-grand, non-cubic maximal clique of size at least $q+4$. Then there exist distinct grand cliques $C$ and $K$ with $M \subseteq C \cup K$ and $C \cap K \neq \varnothing$.*

PROOF. As usual, we assume $0 \in M$. We may further assume that any two distinct non-zero forms $x$ and $y$ of $M$ have distinct radicals (Lemma 2.11), and are of rank two and satisfy $rk(x-y) = 2$ (Lemma 2.10). We first claim that there is a grand clique $C$ with $|C \cap M| \geqslant 4$. Indeed, let $x$ be a fixed non-zero form in $M$. For every non-zero $y$ in $M$, we uniquely determine the grand clique $[0, x, y]$. If $|C \cap M| \leqslant 3$ for all grand cliques $C$ then, in particular, $|[0, x, y] \cap M| = 3$ for all non-zero $y \in M$. As $\{0, x\}$ is contained in precisely $q+1$ grand cliques $C_1, C_2, \ldots, C_{q+1}$, we have $M \subseteq \cup(C_i \cap M)$. But then $|M| \leqslant |\cup(C_i \cap M)| \leqslant 2 + (q+1) = q+3$, a contradiction. This proves the claim.

We next claim we may assume $|C \cap M| \leqslant 4$ for all grand cliques $C$. Indeed, suppose that there exists a grand clique $C$ with $|C \cap M| \geqslant 5$. Translating, if necessary, by a form in $C \cap M$, we may assume that $C \cap M$ contains 0. If there exists a unique form $x \in M$ outside $C$, we may choose $K$ to be any grand clique containing 0 and $x$, and the theorem follows. So assume that there exists two forms $x, y \in M$ lying outside $C$, and consider the grand clique $K = [0, x, y]$. Clearly, $|C \cap K \cap M| \leqslant 2$ as 0 lies in $C \cap K \cap M$, and therefore any two non-zero forms in $C \cap K$ necessarily have the same radical. Thus $C \cap M$ contains at least three non-zero forms, each of which lies outside $K$, and the theorem follows here from Lemmas 3.5 and 2.11.

From above, there exists a grand clique $C$ which satisfies $|C \cap M| = 4$. Again we may assume $0 \in C \cap M$. Choosing $x, y \in M$ with $x, y \notin C$, we consider the grand clique $K = [0, x, y]$. If $|K \cap M| = 3$ then $C \cap K \cap M = \{0\}$, and Lemma 3.5 again applies and we are done. Thus we assume that $|K \cap M| = 4$ and $|C \cap K \cap M| = 2$ for any grand clique $K$ containing 0 and intersecting $M$ in at least two non-zero forms. Let $m$ be the size of the set $X = M \backslash C$. Clearly, there are $\binom{m}{2}$ distinct grand cliques $[0, x, y]$ which can be manufactured from the forms in $X$. (Observe that $z \in [0, x, y]$ is impossible for $z \in X$, $z \neq x, y$, by our size assumptions above.) Moreover, each non-zero form $t$ in $C \cap M$ can occur in at most $[m/2]$ of these cliques; otherwise there would be a form $r \in X$ common to two distinct cliques $[0, t, s]$ and $[0, t, w]$, whence $[0, t, s] = [0, t, r] = [0, t, w]$, a contradiction. We therefore have $\binom{m}{2} \leqslant 3(m/2)$, which implies $m \leqslant 4$. Thus $|M| \leqslant 8 \leqslant q+3$, our final contradiction. The proof of the theorem is now complete. □

THEOREM 6.2. *Let $M$ be a maximal clique of size at least $q+4$. Then $M$ is grand, cubic, quadratic or linear.*

PROOF. Suppose $M$ is neither grand nor cubic. Then, by Theorem 6.1, there exist grand cliques $C_1$ and $C_2$ such that $M \subseteq C_1 \cup C_2$ and $C_1 \cap C_2 \neq \varnothing$. Translating, if necessary, we may assume that $0 \in C_1 \cap C_2$, so all non-zero forms of $M$ are of rank two. Clearly, $M \backslash C_2 \neq \varnothing$. If $|M \backslash C_2| = 1$, then $M = \{x\} \cup N_{C_2}(x)$, where $\{x\} = M \backslash C_2$, and $M$ is quadratic by Lemma 4.6. We therefore assume $|M \backslash C_2| \geqslant 2$ and, by symmetry, $[M \backslash C_1| \geqslant 2$. By Lemma 5.3, $M$ is of the form $(C_1, C_2, C_3)$ for some grand clique $C_3$. Suppose that $M$ is not linear. Without loss of generality, we may assume that $\mathrm{Rad}(r) \subseteq W(C_2)$ for all $r \in C_1 \cap C_3$. Thus $\mathrm{Rad}(r) = W(C_1) \cap W(C_2)$ for all $r \in M \backslash (C_2 \cap C_3)$. By Lemma 2.8, there exists $t \in C_2 \cap C_3$ with $\mathrm{Rad}(t) \neq W(C_1) \cap W(C_2)$, whence $W(C_2) = \mathrm{Rad}(t) \oplus \mathrm{Rad}(z)$ for $z \in C_1 \cap C_2$, $z \neq 0$. But Lemma 2.7 now implies $C_1 \cap C_3 \subseteq C_2$, i.e. $M = (C_1, C_2, C_3) \subseteq C_2$, a contradiction. We conclude that $M$ is linear. □

THEOREM 6.3. *Quadratic cliques are maximal of size $q^2+1$.*

PROOF. Let $R = \{x\} \cup N_C(x)$ be a quadratic clique so that, in particular, $\mathrm{Rad}(x)$ is not contained in $W(C)$. Then $|R| = q^2 + 1$ by Lemma 4.5. Since a linear clique can have size no greater than $3(q+1)$ (Lemma 2.3), the result now follows from Lemma 4.6 and Theorem 6.2. □

We now come to the case of linear cliques. One prefatory remark is in order. We will be using Lemma 2.3 to compute $|C \cap K|$ for two intersecting type 1 cliques $C$ and $K$. If $0 \notin C \cap K$, we will of course translate by a member of $C \cap K$ first. It is possible that one translation will give an elliptic $x_0$ (using the notation of Lemma 2.3), while another may give a hyperbolic $x_0$. What is invariant under translation is whether or not $x_0$ and $y_0$ are of the same type, i.e. both elliptic or both hyperbolic. In view of this, we will call the pair $(C, K)$ pure if $|C \cap K| = q-1$ and mixed if $|C \cap K| = q+1$. A set of pairwise intersecting type 1 cliques will be called pure if every pair is pure; otherwise we shall call it mixed.

THEOREM 6.4. *Linear cliques are maximal of size $3q - \varepsilon$, where $\varepsilon \in \{0, 1, 2, 3, 5\}$.*

PROOF. Let $L$ be a linear clique containing 0 (with notation as in the definition). Clearly, for $z$ non-zero in $C_1 \cap C_2$, we have $\mathrm{Rad}(x_1) \oplus \mathrm{Rad}(x_2) \oplus \mathrm{Rad}(z) = V$, so $L$ cannot be contained in any grand or cubic clique by Lemma 2.9. Let $M$ be a maximal clique containing $L$. Choose $z, z' \in C_1 \cap C_2$, $z, z' \neq 0$. Then $z, z' \in M$ and $\mathrm{Rad}(z) = \mathrm{Rad}(z')$. By Lemma 2.11, there exist distinct grand cliques $K_1$ and $K_2$ such that $M \subseteq K_1 \cup K_2$ and $\{0, z, z'\} \subseteq K_1 \cap K_2$. Without loss of generality, we may assume $x_1 \in K_1$, from which $K_i = [0, z, x_i] = C_i$ easily follows $(i = 1, 2)$. By Lemma 5.3,

$M=(C_1, C_2, K)$ for some grand clique $K$; in particular, $C_i \cap C_3 \subseteq C_i \cap K$ $(i=1,2)$. Thus, from the proof of Lemma 5.3, $K=[0, x_1, x_2]=C_3$ or $K-x_1=[0, x_2-x_1, y_1-x_1]=C_3-x_1$. In either case $K=C_3$, whence $M=L$ as desired.

In order to calculate the size of a linear clique $(C_1, C_2, C_3)$, we need to know $|C_1 \cap C_2 \cap C_3]$ and $|C_i \cap C_j|$ for $1 \leqslant i \neq j \leqslant 3$. We first note that, whenever $C_i$ and $C_j$ are both of type 1, $|C_i \cap C_j| \neq q$. This follows from Lemma 2.3(i) and the existence of two forms $x, y \in C_i \cap C_j$ with $rk(x-y)=2$ (see Lemma 2.10).

The sizes of the various types of linear cliques are shown in Table 1. Non-existence (in cases (c), (g) and (k)) is established below. The sizes in the remaining cases are easily calculated using Lemmas 2.3 and 5.2.

Suppose that $(C_1, C_2, C_3)$ is a linear clique as in (c), with $C_1$ and $C_2$ of type 2, $C_3$ of type 1, and $C_1 \cap C_2 \cap C_3=\varnothing$. Assume that $0 \in C_1 \cap C_2$, and choose a basis $\{v_1, v_2, v_3\}$ of $V$ with $v_1 \in W(C_1)$, $v_2 \in W(C_1) \cap W(C_2)$ and $v_3 \in W(C_2)$. Let $C_3=C(x)$. Since $C_1 \cap C_3 \neq \varnothing$, there is some rank one form $y$ such that $x+y \in C_3$. Then $x|_{W(C_1)}=-y|_{W(C_1)}$, whence $x|_{W(C_1)}$ has rank one. Likewise, $x|_{W(C_2)}$ has rank one. Assuming that $x(v_2) \neq 0$, we may therefore write

$$M_x=\begin{pmatrix} a^2/b & a & d \\ a & b & c \\ d & c & c^2/b \end{pmatrix}.$$

But then $x-y \in C_1 \cap C_2 \cap C_3$, where

$$M_y=\begin{pmatrix} a^2/b & a & ac/b \\ a & b & c \\ ac/b & c & c^2/b \end{pmatrix},$$

contradicting $C_1 \cap C_2 \cap C_3=\varnothing$.

Suppose then that $x(v_2)=0$. In order for $x|_{W(C_1)}$ and $x|_{W(C_2)}$ to both be of rank one, it is necessary that $v_2$ be in $\mathrm{Rad}(x)$. It is now easy to check that $(C_1, C_2, C_3)$ is contained in the cubic clique $C(\langle v_2 \rangle, 0)$. Thus non-existence is established in case (c).

Now let $(C_1, C_2, C_3)$ be a linear clique with $C_1$ and $C_3$ of type 1 and $C_1 \cap C_2 \cap C_3=\varnothing$. We will show that $|C_1 \cap C_3|=q-1$, establishing non-existence in cases (g) and (k). Assume that $0 \in C_1 \cap C_2$. We have two possibilities to consider. In the first, some pair of forms in $C_1 \cap C_3$ have the same radical. The argument in the first paragraph of the

TABLE 1
Sizes of linear cliques

| | Description of grand cliques | $\lvert C_1 \cap C_2 \cap C_3\rvert$ | $\lvert (C_1, C_2, C_3)\rvert$ |
|---|---|---|---|
| (a) | All type 2 | 0 | $3q$ |
| (b) | All type 2 | 1 | $3q-2$ |
| (c) | One type 1 | 0 | Does not exist |
| (d) | One type 1 | 1 | $3q-2$ |
| (e) | Two type 1 (pure) | 0 | $3q-1$ |
| (f) | Two type 1 (pure) | 1 | $3q-3$ |
| (g) | Two type 1 (mixed) | 0 | Does not exist |
| (h) | Two type 1 (mixed) | 1 | $3q-1$ |
| (i) | All type 1 (pure) | 0 | $3q-3$ |
| (j) | All type 1 (pure) | 1 | $3q-5$ |
| (k) | All type 1 (mixed) | 0 | Does not exist |
| (l) | All type 1 (mixed) | 1 | $3q-1$ |

proof of Lemma 5.3 shows, in this case, that $0 \in C_1 \cap C_2 \cap C_3$. So assume no two forms of $C_1 \cap C_3$ have the same radical. Since each radical is in $W(C_1)$, we have $q+1$ distinct radicals possible. If $|C_1 \cap C_3|$ is not $q-1$, then all must be used. One cannot be, however: $W(C_1) \cap W(C_2)$. For suppose $x \in C_1 \cap C_3$ with $\mathrm{Rad}(x) = W(C_1) \cap W(C_2)$. By Lemma 5.1, there exist two forms in $C_2 \backslash C_1$ with distinct radicals. Therefore Lemma 2.7 applies and $x \in C_1 \cap C_2 \cap C_3$, a contradiction. Thus non-existence is proved in each of cases (g) and (k).

Finally, we remark that it is not difficult to construct examples which prove existence in all of the remaining cases. □

We now state our main result.

Corollary 6.5. *Let $M$ be a maximal clique of* Quad$(d, q)$, *where $q \geq 5$ is odd and $|M| \geq q+4$. Then $M$ is either grand, cubic, quadratic or linear. Moreover, all grand, cubic, quadratic and linear cliques are in fact maximal of respective sizes $q^d$, $q^3$, $q^2+1$ and $3q-\varepsilon$, where $\varepsilon \in \{0, 1, 2, 3, 5\}$.*

Proof. This is an immediate consequence of Theorems 6.1–6.4. □

Corollary 6.5 provides the classification of all maximal cliques of size at least $q+4$. The reader may ask: Are there any others? The authors suspect there are. This suspicion is based on the following example of a maximal clique, when $q=3$, which is neither grand, cubic, quadratic nor linear. Not surprisingly, this example fails to satisfy the conclusion of the $q=3$ analogue of Theorem 6.1:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}.$$

## 7. Quadratic Forms and Symplectic Dual Polar Spaces

We are indebted to a referee of this paper for pointing out a connection between the graph Quad$(d, q)$ and the distance 1-or-2 graph of a subgraph of the symplectic dual polar graph $C_d(q)$.

The vertices of $C_d(q)$ are the maximal totally isotropic subspaces of a fixed $(2d)$-dimensional symplectic space (so each vertex is $d$-dimensional). Two vertices are adjacent if they intersect in a $(d-1)$-dimensional space. Now fix a vertex $u$ of $C_d(q)$. We are interested in the subgraph $H = H(u)$ of $C_d(q)$ which is induced on those vertices $w$ which are of distance $d$ from $u$ (equivalently, disjoint from $u$ as vector space). By a construction attributed to Kantor (see [5]), Quad$(d, q)$ is isomorphic to the distance 1-or-2 graph $H^*$ of $H$. (Two vertices are adjacent in $H^*$ iff they are of distance 1 or 2 in $H$.) Thus a maximal clique of Quad $(d, q)$ corresponds to a maximal $\{0, 1, 2\}$-clique, to use Delsarte's notation, of $H$. Below we give a brief description of the $\{0, 1, 2\}$-cliques in $H$ which correspond to the known cliques of Quad$(d, q)$. For $x, y \in C_d(q)$, we denote by $d(x, y)$ the distance from $x$ to $y$ in the graph $C_d(q)$. (Further information on dual polar graphs and dual polar spaces can be found in [1], [3], [4].)

1. *Grand cliques*: *Let $w$ be a vertex of $C_d(q)$ which has distance $d-1$ or $d$ from $u$. The grand cliques are of the form $\{x \in H \colon d(x, w) \leq 1\}$.*

1A. *If $w \in H$, then the clique corresponds to a grand clique of type* 1.
1B. *If $d(w, u) = d - 1$, then the clique corresponds to a grand clique of type* 2.

For the cubic, quadratic and linear cliques, we may assume, by Lemma 2.1, that the graph $C_d(q)$ is of diameter 3. Here we use the more suggestive language of projective geometry. Thus the vertices of $H$ are planes which are disjoint from $u$. We say that two planes meet (resp., hit) if their intersection is a point (resp., line). We refer to any plane disjoint from $u$ as a $d$-plane.

2. *Cubic cliques: Let $P$ be a point not in $u$. Then the set of all $d$-planes on $P$ corresponds to a cubic clique.*

3. *Quadratic cliques: Let $x$ and $y$ be disjoint planes, where $x$ is a $d$-plane and $y$ fails to hit $u$. Then the set of all $d$-planes which meet $x$ and hit $y$ corresponds to a quadratic clique.*

4. *Linear cliques: Let $x$, $y$ and $z$ be planes which pairwise meet, and each of which fails to hit $u$. Assume further that no three of $x$, $y$, $z$, $u$ share a common point. Then the set of all $d$-planes which hit at least two of $x$, $y$ and $z$ corresponds to a linear clique.*

In the light of recent work of Brouwer and Hemmeter [2] on the classification of maximal $\{0, 1, 2\}$-cliques in $C_d(q)$ (which was undertaken without any thought about the connection to Quad$(d, q)$), we are optimistic that the classification of the maximal cliques of Quad$(d, q)$ can be completed for all odd $q$ in the near future.

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